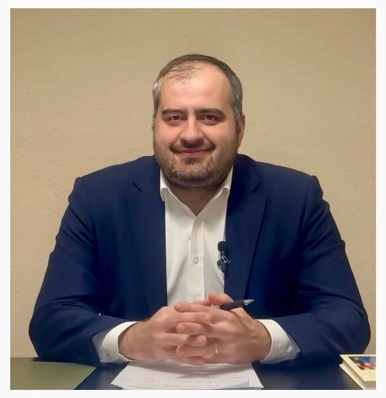
The Resolvent Method, SLE Theory and Random Matrix Theory

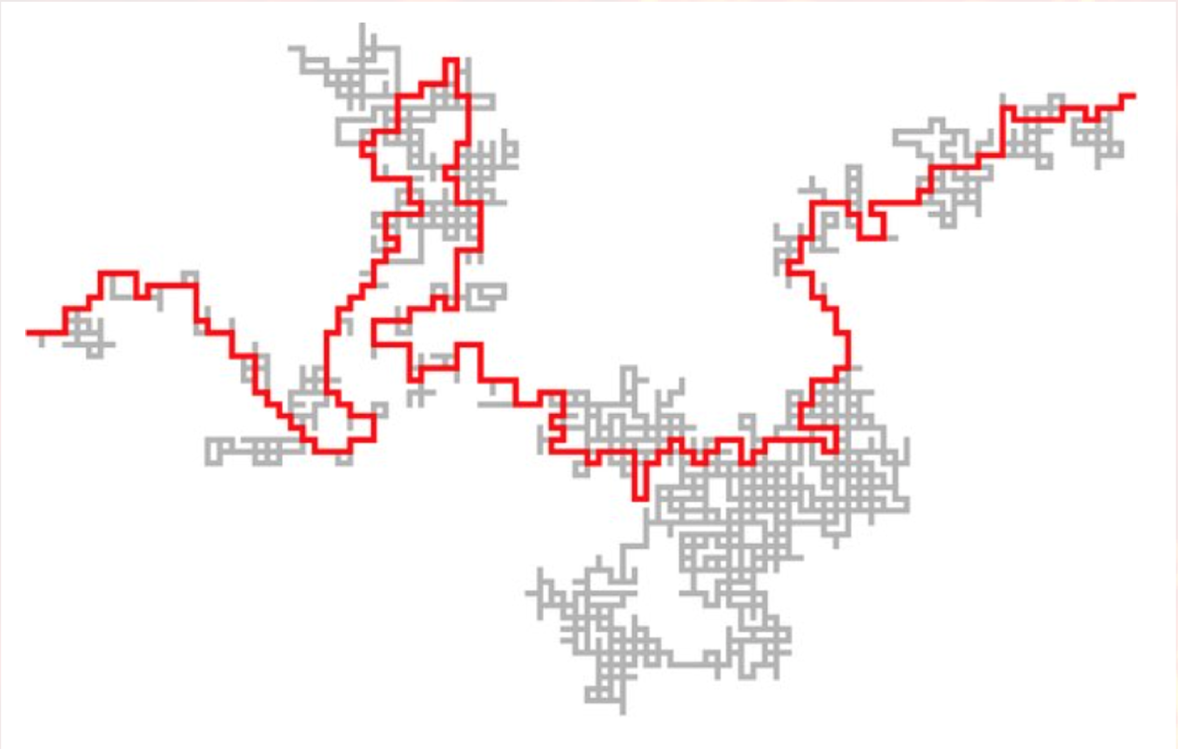
Vlad Margarint
University of North Carolina at Charlotte
margarintvlad.com



Introduction

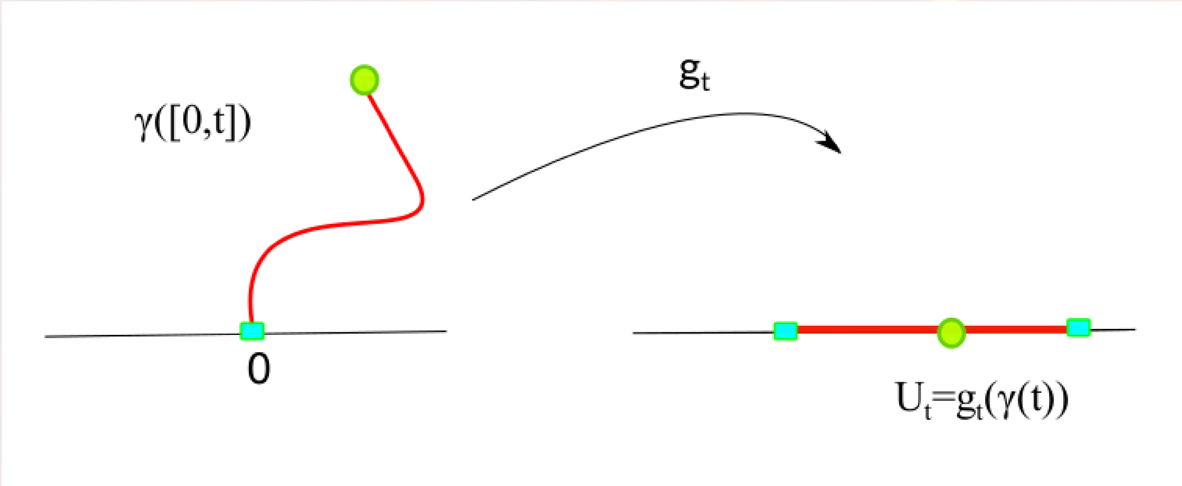
Loewner equation appears in many areas of research, including the study of growth processes in the complex plane, Schramm–Loewner Evolutions (SLE), and more. In this poster, I will describe a toolbox that studies Loewner dynamics via the resolvents of matrices.

- One application of this method consists in the study of the Multiple SLE using modern techniques from Random Matrix Theory (RMT). A very important example of Multiple SLE, due to its connections with Conformal Field Theory (CFT), is the one in which the driver is Dyson Brownian Motion.
- Let us first consider the one-curve model.
- A sample of a planar Loop-Erased Random Walk -introduced by Prof. Greg Lawler:



Credit to E. Peltola, A. Karrila, K. Kytola.

- What is the scaling limit as the size of the lattice tends to zero?
- For a non-self crossing curve $\gamma(t) : [0,\infty) \to \bar{\mathbb{H}}$ with $\gamma(0) = 0$ and $\gamma(\infty) = \infty$, we consider the simply connected domain $\mathbb{H} \setminus \gamma([0,t])$.



- The forward Loewner Differential Equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}.$$

- We refer to the real-valued U_t as the driver of the equation.
- The initial conditions are $g_0(z) = z$.
- SLE curves $\gamma(t) := \lim_{y \to 0+} g_t^{-1}(iy + \sqrt{\kappa} B_t)$. [Rohde-Schramm $\kappa \neq 8$; Lawler-Schramm-Werner $\kappa = 8$.]

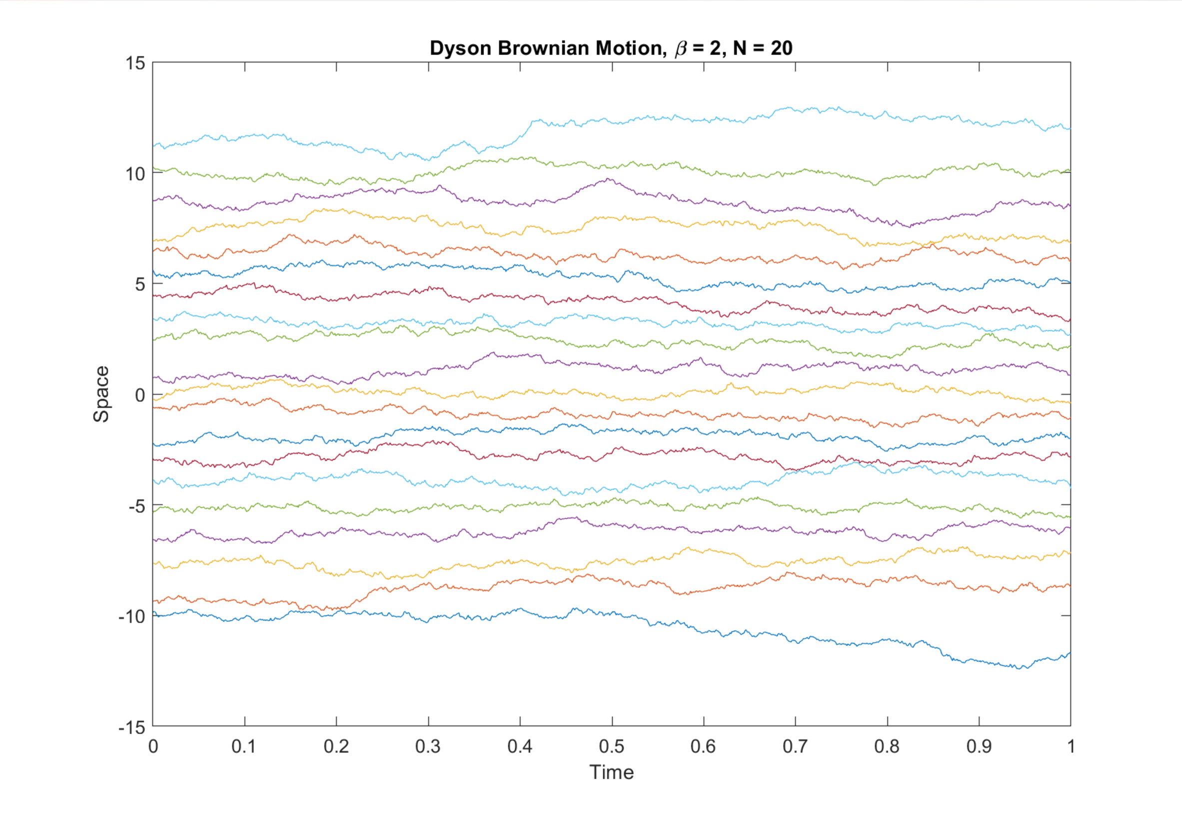
An application of the Resolvent Method: A bridge between Schramm-Loewner Evolutions and Random Matrix Theory

- **We propose to study the model from the Random Matrix Theory viewpoint.**

Let us consider the following:

- **Brownian Motion** $\to$ **Dyson Brownian Motion** (DBM), $\beta = 8/\kappa$.

$$d\lambda_t^i = \frac{\sqrt{2}}{\sqrt{N\beta}} dB_t^i + \frac{1}{N} \sum_{j \neq i} \frac{dt}{\lambda_t^j - \lambda_t^i}, \qquad i = 1, \cdots, N.$$



Credit to A. Swan

- **Multiple SLE** simultaneous growth

$$\partial_t g_t^N(z) = \frac{1}{N} \sum_{j=1}^{N} \frac{2}{g_t^N(z) - \lambda_j^N(t)}.$$

- Feature of DBM: **Very fast** convergence to **local equilibrium**. Locally, the information from the initial conditions is lost very fast.
- This property was used by Erdős, Yau, and collaborators in the three-step-strategy proof of universality in Random Matrices.
- This feature **should impact** the Multiple SLE simultaneous growth.

Theorem 1 (Chen-M.; SPA). *For $N = 2$ drivers, perturbations of initial value/ parameter $\kappa \in (0, 4]$, convergence a.s. / with high probability in the Carathéodory sense.*

Theorem 2 (Campbell-Luh-M.; 2025 RMTA, Featured article). *Let $\beta = 1$ or $\beta = 2$, and let us consider Dyson Brownian motion beginning at the origin. Let K_T be the multiple SLE hull at time $T > 0$. Then, for any $\varepsilon > 0$, for the multiple SLE maps for N curves, we have that*

$$\sup_{t \in [0,T], z \in G} \left| g_t^N(z) - g_t^\infty(z) \right| = O\left(\frac{1}{N^{1/3 - \varepsilon}} \right),$$

with overwhelming probability, for a given $G \subset \mathbb{H} \setminus K_T$.

- Uses RMT techniques such as Stieltjes transforms estimates, self-consistent equations, etc.

The toolbox

- We rewrite the right-hand side of the Multiple Loewner equation as $M_t^N(\bullet) = \frac{1}{N} \sum_{j=1}^{N} \frac{2}{\bullet - \lambda_t^j}$.
- Then, we recognize a Stieltjes transform for a given choice of measure.
- Let $\beta = 1$. We have that $M_t^N(\bullet) = \frac{-2}{N} \operatorname{tr}(A_t - \bullet \cdot I)^{-1}$, where $A_t = \sqrt{t} A$, with A a GOE. Similar for $\beta = 2$.
- Thus, we can recover the right-hand side of the Loewner equation as a Random Matrix Theory object.
- The critical parameters $\kappa = 4$ and $\kappa = 8$ in the SLE theory correspond to the "nice" $\beta = 2$ and $\beta = 1$ parameters in Random Matrix Theory.

A first application of the method

- Let us consider the time interval $[0, 1]$ and a uniform partition with $t_k = \frac{k}{n}, k = 0, 1, \ldots, n$. Let $t \in (t_1, t_2)$.
- The proof of Theorem 2 is based on controlling for $G \subset \mathbb{H}$:

$$\sup_{\bullet \in G} \left| M_t^\infty(\bullet) - M_t^N(\bullet) \right| \leq \sup_{\bullet \in G} \left| M_t^\infty(\bullet) - M_{t_1}^\infty(\bullet) \right| + \sup_{\bullet \in G} \left| M_{t_1}^\infty(\bullet) - M_{t_1}^N(\bullet) \right| + \sup_{\bullet \in G} \left| M_{t_1}^N(\bullet) - M_t^N(\bullet) \right|, \tag{1}$$

- Resolvent estimates control the middle term on the right-hand side.

General view

- For Complex dynamics modeled by the Loewner Equation, we propose an alternative study via the resolvents of 'nicer' matrix dynamics.
- Another example, in addition to our previously studied model, is

$$d\lambda_t^i = \frac{\sqrt{2}}{\sqrt{N\beta}} dB_t^i + \frac{1}{N} \sum_{j \neq i} \frac{dt}{\lambda_t^j - \lambda_t^i} - \frac{\lambda_t^i}{2} dt, \qquad i = 1, \cdots, N.$$

which corresponds in the matrix space to a nicer dynamics of the form

$$dA_t = -\frac{1}{2} A_t dt + dB_t.$$

- For the future, one direction is to study the interaction between the Complex dynamics modeled by the Loewner equation, the matrix dynamics, and the corresponding resolvents in random and deterministic settings.

References

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- J. Chen and V. Margarint. Perturbations of multiple Schramm–Loewner evolution with two non-colliding Dyson Brownian motions. Stochastic Processes and their Applications, 151, 2022.
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